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Ψ- Dichotomy for Linear Dynamic Systems on Time Scales

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Abstract

The intent of this paper is to develop some explicit sufficient criteria for the existence and roughness of exponential Ψ-dichotomies of linear dynamic system of the form $x^{\Delta}(t)=A(t)x(t)$ on time scales. It is more interesting and more challenging to establish necessary and sufficient criteria for the existence of exponential Ψ-dichotomies of dynamic equations on general time scales.

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Introduction

Exponential Ψ-dichotomy generalizes the concept of hyperbolicity from autonomous to non autonomous linear systems, has been playing an ever more important role in the study of non autonomous dynamical systems such as ordinary differential equations , difference equations , and dynamic equations on time scales .

In this paper we develop some explicit necessary and sufficient criteria for the existence of exponential Ψ-dichotomies of linear dynamic system of the form

$$x^{\Delta}(t)=A(t)x(t) \quad (1.1)$$

on time scales. It is more interesting and more challenging to establish necessary and sufficient criteria for the existence of exponential Ψ-dichotomies of dynamic equations on general time scales. The content of this paper is as follows. In Section 2, we introduce some basic preliminary results on the calculus on time scales in order to make this paper self-contained. Section 3 is devoted to establishing explicit necessary and sufficient criteria for the existence of exponential Ψ-dichotomies for linear dynamic equations on time scales.

Preliminaries

In this section, we give a short overview on some basic results on the time scale calculus that are important for the present treatment of exponential Ψ-dichotomies on time scales. For the theory of time scales we refer to the original work by Hilger [5] and to the book by Bohner and Peterson [2].

A Timescale T is a closed subset of R; and examples of time scales include N; Z; R, Fuzzy sets etc. The set Q = $\{t \in R/Q, 0 \le t \le 1\}$ are not time scales. Time scales need not necessarily be connected. In order to overcome this deficiency, we introduce the notion of jump operators. Forward (backward) jump operator σ(t)of t for t < sup T (respectively ρ(t) at t for t >inf T) is given by σ(t) = inf{s ∈ T : s > t} , ρ(t) = sup{s ∈ T : s < t}, for all t ∈ T. The graininess function μ : T → [0,∞) is defined by μ (t) = σ (t) − t. Throughout we assume that T has a topology that it inherits from the standard topology on the real number R. The jump operators σ and ρ allow the classification of points in a time scale in the way: If σ(t) > t, then the point t is called right scattered ; while if ρ(t) < t, then t is termed left scattered. If t < sup T and σ(t) = t, then the point ' t' is called right dense: while if t > inf T and ρ(t) = t, then we say 't' is left-dense. We say that f : T → R is rd-continuous provided f is continuous at each right-dense point of T and has a finite left-sided limit at each left-dense point of T and will be denoted by Crd.

A function f : T → T is said to be differentiable at $t \in T^k = \{T \setminus (\rho(t)\max(T), \max t)\}$

if $\lim_{\sigma(t)\to s} \frac{f((\sigma(t) - f(s))}{\sigma(t) - s}$ where s∈T-{ σ(t)} exist and is said to be differentiable on T provided it is differentiable for each t∈T^k. A function F : T → T, with

F^{Δ} (t)=f(t) for all t∈T^k is said to be integrable, if $\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$ where F is anti derivative of f and for all s, t ∈T. Let f: T → T, and if T=R and a, b ∈T, then f^{Δ} (t)=f'(t) and $\int_a^b f(t)dt = \int_a^b f(t)\Delta t.$

If T=Z, then f^{Δ} (t)= Δf(t)=f(t+1)-f(t) and

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=a}^{b-1} f(k) & if \quad a < b \\ 0 & if \quad a = b \\ \sum_{k=b}^{a-1} f(k) & if \quad a > b \end{cases}$$

If f, g: T → X(X is a Banach space) be differentiable in t∈T^k. Then for any two scalars α, β the mapping α f+β g is differentiable in t and further we have:

1.(α f+β g)$^{\Delta}$(t)= α f^{Δ}(t) +β g^{Δ}(t) 2. (fg)$^{\Delta}$(t) =(f)$^{\Delta}$(t)g(t)+f(σ(t)) g^{Δ}(t)

3. f(σ(t)) =f(t)+ μ (t)f^{Δ}(t) 4. (kf)$^{\Delta}$(t) = k f^{Δ}(t), for any scalar k.

If f is Δ-differentiable, then f is continuous. Also if t is right scattered and f is continuous at t then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} \quad .$$

An n × n-matrix-valued function A(t) on T is called regressive if I + μ(t)A(t) is invertible for all t ∈ T. The set of functions being both regressive and rd-continuous is denoted by
R = **R**(T) = **R**(T, R)(**R**(T, $R^{n\times n}$)). The set of all regressive functions defined on T forms an Abelian group under the addition ⊕ defined by (p ⊕ q)(t) := p(t) + q(t) + μ(t)p(t)q(t) and the additive inverse in this group is given by ⊖p(t) := − p(t)/(1+μ(t)p(t)) .Given a p ∈ R, the exponential function is defined by

$$e_p(t,s) = \begin{cases} \exp(\int_s^t p(\tau)\Delta\Delta\tau) & \mu(t) = 0 \\ \exp(\int_s^t \frac{1}{\mu(t)} \mathrm{Log}(1 + p(\tau)\mu(t)\Delta t)\Delta & \mu(t) \neq 0 \end{cases} \quad \text{for } s, t \in T$$

where Log is the principal logarithm, and has the following properties
$e_p(t, t) \equiv 1$, $e_p(t, s) = 1/ e_p(s, t) = e_{\ominus p}(s, t)$, $e_p(t, s)e_p(s, r) = e_p(t, r)$, $[e_p(\cdot, s)]^{\Delta} = p e_p(\cdot, s)$.

In this paper, T is assumed to be unbounded above and below and
$\vartheta := \min\{[0,\infty) \cap T\}$, $T+ := [\vartheta,\infty) \cap T$,

$\chi := \sup_{t\in T} \mu(t) \in [0,+\infty)$ $|x| := \sup_i |x_i|$, $x \in R^n$

We introducing definitions and notation that will be useful in proving the main results. The Euclidian norm of an n ×1 vector x(t) is defined to be a real valued function of t and is denoted by $\|x(t)\| = \sqrt{x^T(t)x(t)}$. The induced norm of an n×n matrix A is defined to be

$$\|A\| = \max_{\|x\| \le 1} \|Ax\|$$

$Let\,\psi_i : T \to (0,\infty)$, i=1, 2,…n, be rd continuous functions and $\Psi = \mathrm{diag}[\Psi_1, \Psi_2, \text{--------}\Psi_n]$.

Necessary and Sufficient Criteria for Exponential Ψ-Dichotomy

Consider the following linear dynamic equation on time scales

$$x^{\Delta}(t) = A(t)x(t) \qquad (3.1)$$

where $A \in \mathbf{R}$. First, we introduce the notion of exponential Ψ-dichotomies on time scales.

Definition 3.1 ([8]). The dynamical system (3.1) is said to have an exponential Ψ-dichotomy on T, if there exist a projection matrix P (i.e., $P^2 = P$) on R^n and positive constants M_i and α_i, i = 1, 2, such that

$$|\Psi(t)X(t)PX^{-1}(s)\Psi^{-1}(s)| \le M_1 e_{\ominus\alpha 1}(t, s),\ t \ge s,$$
$$|\Psi(t)X(t)(I-P)X^{-1}(s)\Psi^{-1}(s)| \le M_2 e_{\ominus\alpha 2}(s, t),\ t \le s, \qquad (3.2)$$

where X is a fundamental solution matrix of (3.1) and I is the identity matrix. When (3.2) holds with $\alpha_1 = \alpha_2 = 0$, (3.1) is said to possess an ordinary Ψ-dichotomy.

Remark 3.1. We can choose an appropriate fundamental solution matrix such that the projections P and I−P can be written as

$I_{ko} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, I_{o(n-k)} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix}$, respectively, where I_k is a k × k identity matrix and I_{n-k} is an (n − k) × (n − k) identity matrix. In fact, there exists a nonsingular matrix B such that $P = BI_{k0}B^{-1}$, then (3.2) reduces to

$$|\Psi(t)X(t)BI_{k0}B^{-1}X^{-1}(s)\Psi^{-1}(s)| \le M_1 e_{\ominus\alpha 1}(t, s),\ t \ge s,$$
$$|\Psi(t)X(t)BI_{0(n-k)}B^{-1}X^{-1}(s)\Psi^{-1}(s)| \le M_2 e_{\ominus\alpha 2}(s, t),\ t \le s.$$

Let $X_0(t) = X(t)B$. Then it is easy to show that X_0 is also a fundamental solution matrix.
In addition, we also obtain the following fact in (3.2). If $\chi > 0$, then for any $x \in (0, \chi]$ and $\alpha > 0$, $f_1(x) := (1/x)\log(1/(1+\alpha x))$ is strictly increasing with $\lim_{x\to 0+} = f_1(x) = -\alpha$
and $f_2(x) := (1/x)\log(1 + \alpha x)$ is strictly decreasing satisfying $\lim_{x\to 0+} f_2(x) = \alpha$.

Therefore, for t ≥ s, we have

$$e^{\alpha(t-s)} \ge e_\alpha(t, s) \ge (1 + \alpha\chi)^{t-s/\chi}, \qquad e^{-\alpha(t-s)} \le e_{\ominus\alpha}(t, s) \le (1/1 + \alpha\chi)^{(t-s)/\chi} \qquad (3.3)$$

Lemma 3.1. The dynamical system (3.1) has an exponential Ψ-dichotomy on T if the following conditions are satisfied:
(i) There exist positive constants L_i and α_i (i = 1, 2) such that

$$|\Psi(t)X(t)P\xi| \le L_1 e_{\ominus\alpha 1}(t, s)|\Psi(s)X(s)P\xi|,\ t \ge s,$$
$$|\Psi(t)X(t)(I-P)\xi| \le L_2 e_{\ominus\alpha 2}(s, t)|\Psi(s)X(s)(I-P)\xi|,\ t \le s, \qquad (3.4)$$

where ξ is an arbitrary n-dimensional vector;
(ii) The dynamical system (3.1) has bounded growth, that is, there exist K ≥ 1 and β > 0 such that

$$|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)| \le Ke_\beta(t, s),\ t \ge s. \qquad (3.5)$$

The following theorem represents a useful property of the exponential Ψ- dichotomy on time scales.

Theorem 3.1. If the dynamical system (3.1) has an exponential Ψ-dichotomy on $[t_0,\infty) \in T$ for some fixed $t_0 \geq \vartheta$, then it has also an exponential Ψ-dichotomy on T+ with the same projection P and the same exponents α_1, α_2.
Proof. Choose an $K_1 \geq 1$ such that $K_1 \geq e_{|A|}(t_0, \vartheta)$. Then we have
$|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)| \leq K_1$ for $\vartheta \leq s$, $t \leq t_0$. To obtain the conclusions, we consider the following two cases:
Case 1: If $\vartheta \leq s \leq t_0 \leq t$, then

$$|\Psi(t)X(t)PX^{-1}(s)\Psi^{-1}(s)| \leq K_1|\Psi(t)X(t)PX^{-1}(t_0)\Psi^{-1}(t_0)| \leq K_1 M_1 e_{\ominus\alpha_1}(t, t_0) = K_1M_1 e_{\ominus\alpha_1}(t, s)\, e_{\ominus\alpha_1}(s, \vartheta)\, e_{\ominus\alpha_1}(\vartheta, t_0) \leq K_1M_1e_{\alpha_1}(t_0, \vartheta)\, e_{\ominus\alpha_1}(t, s);$$

Case 2: If $\vartheta \leq s \leq t \leq t_0$, then

$$|\Psi(t)X(t)PX^{-1}(s)\Psi^{-1}(s)| \leq K^2_1|\Psi(t_0)X(t_0)PX^{-1}(t_0)\Psi^{-1}(t_0) \leq K^2_1M_1 \leq K^2_1M_1e_{\alpha_1}(t_0, t)= K^2_1M_1 e_{\ominus\alpha_1}(t, s)\, e_{\ominus\alpha_1}(s, \vartheta)\, e_{\ominus\alpha_1}(\vartheta, t_0) \leq K^2_1M_1e_{\alpha_1}(t_0, \vartheta)\, e_{\ominus\alpha_1}(t, s).$$

Therefore,

$$|\Psi(t)X(t)PX^{-1}(s)\Psi^{-1}(s)| \leq M^*_1\, e_{\ominus\alpha_1}(t, s), \quad \vartheta \leq s \leq t, \text{ where } M^*_1 = K^2_1M_1e_{\alpha_1}(t_0, \vartheta).$$

Similarly, we have

$$|\Psi(t)X(t)PX^{-1}(s)\Psi^{-1}(s)|\, | \leq M^*_2\, e_{\ominus\alpha_2}(s, t), \quad \vartheta \leq t \leq s, \text{ where } M^*_2 = N^2_1M_2e_{\alpha_2}(t_0, \vartheta).$$

Using these theorems we develop some explicit necessary and sufficient criteria for the linear dynamic equation (3.1) to have an exponential Ψ -dichotomy.

Theorem 3.2. Assume that $A \in \mathbf{R}$ is bounded. The Linear dynamical system (3.1) has an exponential Ψ -dichotomy on T^+ if and only if there exist positive constants $0 < \theta < 1$, $T > 0$ such that any solution x(t) of (3.1) satisfies

$$|\Psi(t)x(t)| \leq \theta \sup_{|\tau-t|\leq T} |\Psi(\tau)x(\tau)|, \quad t \geq T \tag{3.6}$$

Proof : Suppose the equation (3.1) has an exponential Ψ -dichotomy on T^+, then it follows from Lemma 3.1 that (3.4) holds on T+.
Let x (t) be any solution of (3.1) and set
$x_1(t) = X(t)PX^{-1}(t)x(t)$, $x_2(t) = X(t)(I - P)X^{-1}(t)\, x(t)$, then
$x(t) = X(t)PX^{-1}(s)\, x_1(s) + X(t)(I - P)X^{-1}(s)\, x_2(s)$.
Consider the following two cases:
Case 1: If $|\Psi(s)x_2(s)| \geq |\Psi(s)x_1(s)|$, then, for $t \geq s$, we have
$|\Psi(t)x(t)| \geq |\Psi(t)X(t)(I - P)X^{-1}(s)x_2(s)| - |\Psi(t)X(t)PX^{-1}(s)\, x_1(s)|$.
By the second inequality of (3.4), we have
$|\Psi(t)X(t)(I - P)\xi| \geq L^{-1}_2 \Psi(s)|X(s)(I - P)\xi|\, e_{\alpha_2}(t, s)$ for $t \geq s \geq \vartheta$.
Choosing $\xi = X^{-1}(s)\, x_2(s)$, for $t \geq s \geq \vartheta$, we obtain

$$|\Psi(t)X(t)(I - P)X^{-1}(s)\, x_2(s)| \geq L^{-1}_2 |\Psi(s)X(s)(I - P)X^{-1}(s)x_2(s)|e_{\alpha_2}(t, s) = L^{-1}_2 |x_2(s)|\, e_{\alpha_2}(t, s).$$

For sufficiently large t, it is easy to show that

$$|\Psi(t)x(t)| \geq L^{-1}_2\, e_{\alpha_2}(t, s)\, |\Psi(s)x_2(s)| - L_1 e_{\ominus\alpha_1}(t, s)\, |\Psi(s)x_1(s)| \geq (L^{-1}_2\, e_{\alpha_2}(t, s) - L_1 e_{\ominus\alpha_1}(t, s))\, |\Psi(s)x_2(s)| \geq (1/2)(L^{-1}_2\, e_{\alpha_2}(t, s) - L_1 e_{\ominus\alpha_1}(t, s))\, |\Psi(s)x(s)|.$$

Case 2: If $|\Psi(s)x_1(s)| \geq |\Psi(s)x_2(s)|$, similarly, for $s \geq t \geq \vartheta$, we get
$|\Psi(t)x(t)| \geq (1/2)(L^{-1}_1\, e_{\alpha_1}(s, t) - L_2\, e_{\ominus\alpha_2}(s, t))\, |\Psi(s)x(s)|$.
This means that there exist $0 < \theta < 1$ and $T > 0$ such that

$$L^{-1}_2\, e_{\alpha_2}(\tau + T, \tau) - L_1 e_{\ominus\alpha_1}(\tau + T, \tau) \geq 2\theta^{-1},$$
$$L^{-1}_1\, e_{\alpha_1}(\tau + T, \tau) - L_2 e_{\ominus\alpha_2}(\tau + T, \tau) \geq 2\theta^{-1}.$$

Then $|\Psi(t)x(t)| \leq \theta \,|\sup_{|\tau-t|\leq T} \Psi(\tau)x(\tau)|$, $t \geq T$.

Conversely assume that (3.6) holds. We first show that there exists a constant $c > 1$ such that
$|\Psi(t)x(t)| \leq c|\Psi(s)x(s)|$ for $\vartheta \leq s \leq t \leq s + T$, where x(t) is any nontrivial solution of (3.1).

According to the condition, there exists an $N > 0$ such that $|A(t)| \le N$ for any $t \in T$. It is easy to show that $|\Psi(t)X(t)X^{-1}(s)\xi| \le e_M(t, s)|\xi|$ for $t \ge s$. Let $\xi = \Psi(s)X(s)\xi^*$. For $\vartheta \le s \le t \le s+T$,
we have $|\Psi(t)X(t)\xi^*| \le e_N(s + T, s)|\Psi(s)X(s)\xi^*| \le e^{NT}|\Psi(s)X(s)\xi^*|$, that is,
$|\Psi(t)x(t)| \le c|\Psi(s)x(s)|$, where $c = e^{NT}$.
Suppose that $x(t)$ is a nontrivial bounded solution of (3.1).
Set $\pi(s) = \sup_{\tau \ge s} |\Psi(\tau)x(\tau)|$ for $s \ge \vartheta$, we have

$$|\Psi(t)x(t)| \le \theta \sup_{|\tau - t| \le T} |\Psi(\tau)x(\tau)| \le \theta\pi(s), \quad t \ge s + T.$$

Hence $|\pi(s)| = \sup_{s \le \tau \le s+T} |\Psi(\tau)x(\tau)|$, which implies that

$$|\Psi(t)x(t)| \le c|\Psi(s)x(s)|, \ \vartheta \le s \le t < \infty.$$

If $s + nT \le t \le s + (n + 1)T$, then

$$|\Psi(t)x(t)| \le \theta^n \sup_{|\tau - t| \le nT} |\Psi(\tau)x(\tau)| \le \theta^n c|\Psi(s)x(s)| \le \theta^{-1} c\theta^{(t-s)/T}|\Psi(s)x(s)|.$$

Set $K = \theta^{-1}c$ and $\alpha = -(1/T)\log\theta$. Then we get
$|\Psi(t)x(t)| \le Ne^{-\alpha(t-s)}|\Psi(s)x(s)| \le Ne_{\ominus\alpha}(t, s)|\Psi(s)x(s)|, \quad \vartheta \le s \le t < \infty.$
Carrying out arguments similar to those in Proposition 2.1 in [3], it is easily show that there exists a $T^* > \vartheta$ such that
$|\Psi(t)x(t)| \le Ne_{\ominus\alpha}(s, t)|\Psi(s)x(s)|$ for $T^* \le t \le s < \infty$.
Since A is bounded, then (3.1) has Ψ-bounded growth. From Lemma 3.1 and Theorem3.1, The equation (3.1) has an exponential Ψ-dichotomy on$T+$.

Now we discuss the relationship between the exponential Ψ- dichotomy of the linear dynamic equation (3.1) and the Ψ- bounded solutions of the inhomogeneous linear system corresponding to (3.1). Some necessary and sufficient conditions are derived for (3.1) to have an exponential Ψ-dichotomy.

Consider the following inhomogeneous linear dynamic equation on time scales

$$x^{\Delta}(t) = A(t)x(t) + f(t) \qquad (3.7)$$

where $A \in R$, $f \in C_{rd}(T)$.
Define

$$C_\Psi = \{f \in C_{rd}(T) : \|f\|_{C\Psi} = \sup_{t \in T^+} |\Psi(t)f(t)|\},$$

$$D_\psi = \left\{ f \in C_{rd}(T) : \|f\|_{D_\psi} = \int_{\vartheta}^{\infty} |\psi(\tau)f(\tau)\Delta\tau| \right\}$$

$$E_\psi = \left\{ f \in C_{rd}(T) : \|f\|_{E_\psi} = \sup_{t \in T^+} \frac{1}{\omega} \int_t^{t+\omega} |\psi(\tau)f(\tau)\Delta\tau| \quad \text{where T is } \omega\text{-periodic with } \omega > 0 \right\}$$

here C_Ψ, D_Ψ and E_Ψ are all the Banach spaces.

Lemma 3.2. If $g \in E_\Psi$ is a non-negative function with $\frac{1}{\omega}\int_t^{t+\omega} \psi(\tau)g(\tau)\Delta\tau \le K_2$ for all $t \ge \vartheta$, then

$$\int_{\vartheta}^{t} e_{\Theta\alpha 1}(t,\sigma(\tau))\psi(\tau)g(\tau)\Delta\tau \le \frac{K_2\omega(1+\alpha_1\chi)}{1-e_{\Theta\alpha_1}(\vartheta+\omega,\vartheta)} \qquad \int_{t}^{\infty} e_{\Theta\alpha 2}(\sigma(\tau),t)\psi(\tau)g(\tau)\Delta\tau \le \frac{K_2\omega}{1-e_{\Theta\alpha_2}(\vartheta+\omega,\vartheta)}$$

hold for α_1, $\alpha_2 > 0$ and $t \ge \vartheta$.

The following lemma will be very useful. We first assume that U_1 is the subspace of R^n consisting of the initial values of all Ψ- bounded solutions of (3.1), and U_2 is any fixed subspace of R^n supplementary to U_1 such that R^n can be written as the direct sum
$R^n = U_1 \oplus U_2$.

Lemma 3.3. If (3.7) has a Ψ- bounded solution for $f \in B_\Psi$, where B_Ψ denotes any one of the Banach spaces C_Ψ, D_Ψ and E_Ψ then there exists a positive constant rB_Ψ such that, for every $f \in B_\Psi$, the unique Ψ- bounded solution z(t) of (3.7) with $z(\vartheta) \in U_2$ satisfies
$\|z\|_{C\Psi} \le r_B \| f \|_{B\Psi}$.

Theorem 3.3. Assume that $A \in$ **R** is bounded. Then (3.1) has an ordinary Ψ -dichotomy on T^+ if and only if (3.7) has at least one Ψ -bounded solution for every $f \in D_\Psi$.

Proof: Assume that (3.1) has an ordinary Ψ -dichotomy on T^+. Then it is easy to show that

$$x(t) = \int_{\vartheta}^{t} \psi(t)X(t)PX^{-1}(\sigma(s))f(\tau)\Delta\tau - \int_{t}^{\infty} \psi(t)X(t)(I-P)X^{-1}(\sigma(s))f(\tau)\Delta\tau \qquad (3.8)$$

is a solution of (3.7) and $|\Psi(t)x(t)| \le \max\{M_1, M_2\}\|f\|_{D\Psi}$ for all $t \in T^+$.

Conversely suppose that (3.7) has at least one Ψ -bounded solution for every $f \in D_\Psi$. Set

$$H(t,s) = \begin{cases} \psi(t)X(t)X^{-1}(s) \; for t > s \ge \vartheta \\ -\psi(t)X(t)(I-P)X^{-1}(s) \; for s > t \ge \vartheta \end{cases}$$

where X(t) is a fundamental solution matrix of (3.1) with $X(\vartheta) = I$.

Let $z(t) = \int_{\vartheta}^{\infty} H(t,\sigma(t))f(\tau)\Delta\tau$. For a fixed $t_1 \in T+$, choose a function $f \in D_\Psi$ which vanishes for $t \ge t_1$. Since

$$\psi(t)z(t) = \psi(t)X(t)P\int_{\vartheta}^{t_1} X^{-1}(\sigma(\tau))f(\tau)\Delta\tau; t \ge t_1$$

and

$z(\vartheta) = -(I-P)\int_{\vartheta}^{t_1} X^{-1}(\sigma(\tau))f(\tau)\Delta\tau \in U_2$ then $z(t) = \int_{\vartheta}^{t_1} H(t,\sigma(t))f(\tau)\Delta\tau.$ is Ψ -bounded solution of (3.7). By Lemma 3.3, we have $\|z\|_C \le r_L\|f\|_{D_\psi}$. For any fixed point $s \in T^+$.

we have three cases as in the following: (1) s is right-dense; (2) s is both right-scattered and left-scattered; (3) s is right-scattered and left-dense.

Then

$$|\Psi(t)X(t)PX^{-1}(s)\Psi^{-1}(s)| \le r_{D\Psi}(1 + \chi\|A\|_{C\Psi}) \text{ for } t > s,$$
$$|\Psi(t)X(t)(I-P)X^{-1}(s)\Psi^{-1}(s)| \le r_{D\Psi}(1 + \chi\|A\|_{C\Psi}) \text{ for } s < t. \qquad (3.9)$$

From the continuity of Ψ (t)X(t), it follows that (3.9) is also valid for s = t.

Theorem 3.4. Assume that (3.1) has Ψ-bounded growth. Then (3.1) has an exponential Ψ -dichotomy on T^+ if and only if (3.7) has at least one Ψ-bounded solution for every $f \in C_\Psi$.

Proof: Assume that (3.1) has an exponential Ψ-dichotomy on T^+. Then (3.8) is a solution of (3.7) and

$$|\psi(t)x(t)| = \|f\|_{C_\psi} \left(\int_{\vartheta}^{t} |\psi(t)X(t)PX^{-1}(\sigma(\tau))|\Delta\tau + \int_{t}^{\infty} |\psi(t)X(t)(I-P)X^{-1}(\sigma(\tau))|\Delta\tau\right)$$

$$\leq \|f\|_{C_\psi}\left(M_1\int_{\vartheta}^{t} e_{\Theta\alpha 1}(t,\sigma(\tau))\Delta\tau + M_2\int_{t}^{\infty} e_{\Theta\alpha 2}(\sigma(\tau),t)\Delta\tau\right)$$

$$\leq \|f\|_{C_\psi}\left(\frac{M_1(1+\alpha_1\chi)}{\alpha_1} + \frac{M_2}{\alpha_2}\right).$$

Conversely suppose that (3.7) has at least one Ψ-bounded solution for every function f ∈ C_Ψ. For a fixed q∈ T^+, choose a rd-continuous function η such that 0 ≤ η(t) ≤ 1 for all t ≥ ϑ and
η (t) = 0 for t ≥q. Set f (t) = η (t)x(t)| Ψ (t)x(t)|$^{-1}$, where Ψ (t)x(t) = Ψ (t)X(t)ξ is any nontrivial solution of (3.7). Clearly ‖ f $\|_{C\Psi}$ ≤ 1. Implies

$$\int_{t_0}^{q} H(t,\tau)x(\tau)|x(\tau(\tau)\psi(|^{-1}\Delta\tau \leq r_C, \quad \text{for} \quad \vartheta \leq t_0 \leq q \quad \text{and} \quad t \geq \vartheta$$

If q=t for t=t_1 then

$$\|\psi(t)X(t)P\xi\| \int_{t_0}^{t} |\psi(\tau)X(\tau)P\xi|^{-1}\Delta\tau \leq r_C \quad \text{for} \quad t \geq t_0 \geq \vartheta \tag{3.10}$$

$$\|\psi(t)X(t)(I-P)\xi\| \int_{t}^{q} |\psi(\tau)X(\tau)(I-P)\xi|^{-1}\Delta\tau \leq r_C \quad \text{for} \quad t \geq t_0 \geq \vartheta$$

Replacing ξ by Pξ or (I-P)ξ, we get

$$\int_{t_0}^{s} |\psi(\tau)X(\tau)P\xi|^{-1}\Delta\tau \leq e_{\Theta r_C^{-1}}(t,s)\int_{t_0}^{t} |\psi(\tau)X(\tau)P\xi|^{-1}\Delta\tau \quad \text{for} \quad t \geq s \geq t_0 \tag{3.11}$$

$$\int_{s}^{q} |\psi(\tau)X(\tau)(I-P)\xi|^{-1}\Delta\tau \leq e_{\Theta r_C^{-1}}(s,t)\int_{t}^{q} |\psi(\tau)X(\tau)(I-P)\xi|^{-1}\Delta\tau \quad \text{for} \quad t \leq s \leq q.$$

According to the condition, (3.1) has Ψ-bounded growth, then there exist a K ≥ 1 and a β > 0 such that | Ψ(t)X(t)X^{-1}(s) Ψ^{-1}(s)| ≤ Ke_β (t, s) for t ≥ s. Assume that x is any solution of (3.1) and let $x_1(t) = X(t)PX^{-1}(s)x(s)$, $x_2(t) = X(t)(I - P)X^{-1}(s)x(s)$.

Next we show that $|x_1(t)| \leq e_K|x(s)|\, e_{\Theta r_C^{-1}}(t, s)$ for s ≤ t < ∞ if | Ψ(t)x_1(t)| ≤ K| Ψ(s)x(s)| for some fixed s ≥ ϑ and s ≤ t ≤ s + r_C .

Let t*=inf{t∈ T^+/t≥s+r_C} since x is a solution of (3.1) then x(t)= Ψ(t)X(t) ξ. Replacing t_0 by s and s by t* in the first inequality of (3.11), we obtain

$$\frac{r_C}{K|\psi(s)x(s)|} \leq \int_{s}^{t^+} |\psi(\tau)x_1(\tau)|^{-1}\Delta\tau \leq e_{\Theta r_C^{-1}}(t,s)\int_{s}^{t} |\psi(\tau)x_1(\tau)|^{-1}\Delta\tau \qquad t \geq s + r_C.$$

By the first inequality of (3.10), we have

$$|\psi(t)x_1(t)| \leq r_C\left(\int_{s}^{t}(|\psi(\tau)x(\tau)|^{-1}\Delta\tau\right)^{-1} \leq e_K|\psi(s)x(s)|e_{\Theta r_C^{-1}}(t,s) \qquad t \geq s + r_C$$

Note that

$$e_{\Theta r_C^{-1}}(t,s) \ge e^{\Theta r_C^{-1}(t-s)} \ge 1; s \le t \le s + r_C$$

This implies that

$$|\psi(t)x_1(t)| \le e_K |\psi(s)x(s)| e_{\Theta r_C^{-1}}(t,s) \quad s \le t < \infty \qquad (3.12)$$

Similarly, if $|\Psi(t)x_2(t)| \le K|\Psi(s)x(s)|$ for some fixed $s \ge \vartheta$ and $\max\{\vartheta, s-r_C\} \le t \le s$. we have

$$|x_2(t)| \le e_K |x(s)| e_{\Theta r_C^{-1}}(s,t), \quad \vartheta \le t < s \qquad (3.13)$$

Replacing ξ by $X^{-1}(s)\,\Psi^{-1}(s)\,\xi$ and putting $t \to \infty$ in the second inequality of (3.11),we get

$$\left|\psi(t)X(t)(I-P)X^{-1}(s)\psi^{-1}(s)\xi\right| \le r_C\Big(\int_s^\infty\big(\left|\psi(\tau)X(\tau)X^{-1}(s)\psi^{-1}(s)\right|\Delta\tau\big)^{-1}$$

$$\le r_C[K^{-1}|\xi|^{-1}\int_s^\infty e_\beta(s,\tau)\Delta\tau]^{-1}, \qquad t \le s$$

Since ξ is an arbitrary, we obtain

$$\left|\psi(t)X(t)(I-P)X^{-1}(s)\psi^{-1}(s)\right| \le r_C\beta K, \qquad t \le s.$$

similarly

$$\left|\psi(t)X(t)(I-P)X^{-1}(s)\psi^{-1}(s)\right| \le r_C\beta K e_\beta(t,s), t \ge s.$$

then

$$\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le (1+r_C\beta)Ke_\beta(t,s), t \ge s.$$

Let t_0=s, then by the first inequality of (3.11), we have

$$\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le r_C\beta K[1-e_{\Theta\beta}(t,s)]^{-1}, t \ge s.$$

Now we consider the two cases

(1)As in [3] If $\chi = 0$, then $\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le (1+2r_C\beta)K, t \ge s.$

(2)If $\chi > 0$, it follows from (3.3) that

$$\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le r_C\beta K[1-(\frac{1}{(1+\beta\chi)})^{t-s/\chi}]^{-1}, t > s.$$

Then we get

$$\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le r_C K(1+\beta\chi)e_\beta(t,s) \le K(1+r_C\beta)e^{\beta(t-s)}$$

$$\le K(1+r_C\beta)e^{\beta\chi}.$$

Hence we have

$$\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le \max\left\{\frac{r_C K(1+\beta\chi)}{\chi}, K(1+r_C\beta)e^{\beta\chi}\right\}, t \ge s$$

Define

$$N(\chi)=\begin{cases}(1+2r_C\beta)K & \text{if} \quad \chi = 0.\\ \max\left\{\frac{r_C K(1+\beta\chi)}{\chi}, K(1+r_C\beta)e^{\beta\chi}\right\}, & \text{if}\ \chi > 0.\end{cases}$$

Then

$\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le N(\chi)$ for $t \ge s.$ It follows from(3.12) and(3.13) that

$$\left|\psi(t)X(t)PX^{-1}(s)\psi^{-1}(s)\right| \le e_{N(\chi)} e_{\Theta r_c^{-1}}(t,s) \text{ for } t \ge s \ge \vartheta.$$

$$\left|\psi(t)X(t)(I-P)X^{-1}(s)\psi^{-1}(s)\right| \le e_{r_C\beta K} e_{\Theta r_c^{-1}}(s,t) \text{ for } s \ge t \ge \vartheta.$$

This implies that (3.1) has an exponential Ψ -dichotomy on T^+.

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